

## Hw 2 Solutions

### Problem 1.5.2 Solution

Let  $s_i$  denote the outcome that the roll is  $i$ . So, for  $1 \leq i \leq 6$ ,  $R_i = \{s_i\}$ . Similarly,  $G_j = \{s_{j+1}, \dots, s_6\}$ .

- (a) Since  $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$  and all outcomes have probability  $1/6$ ,  $P[G_1] = 5/6$ . The event  $R_3G_1 = \{s_3\}$  and  $P[R_3G_1] = 1/6$  so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \quad (1)$$

- (b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \quad (2)$$

- (c) The event  $E$  that the roll is even is  $E = \{s_2, s_4, s_6\}$  and has probability  $3/6$ . The joint probability of  $G_3$  and  $E$  is

$$P[G_3E] = P[s_4, s_6] = 1/3. \quad (3)$$

The conditional probabilities of  $G_3$  given  $E$  is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \quad (4)$$

- (d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \quad (5)$$

### Problem 1.5.3 Solution

Since the 2 of clubs is an even numbered card,  $C_2 \subset E$  so that  $P[C_2E] = P[C_2] = 1/3$ . Since  $P[E] = 2/3$ ,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \quad (1)$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \quad (2)$$

### Problem 1.6.4 Solution

(a) Since  $A \cap B = \emptyset$ ,  $P[A \cap B] = 0$ . To find  $P[B]$ , we can write

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \quad (1)$$

$$5/8 = 3/8 + P[B] - 0. \quad (2)$$

Thus,  $P[B] = 1/4$ . Since  $A$  is a subset of  $B^c$ ,  $P[A \cap B^c] = P[A] = 3/8$ . Furthermore, since  $A$  is a subset of  $B^c$ ,  $P[A \cup B^c] = P[B^c] = 3/4$ .

(b) The events  $A$  and  $B$  are dependent because

$$P[AB] = 0 \neq 3/32 = P[A]P[B]. \quad (3)$$

(c) Since  $C$  and  $D$  are independent  $P[CD] = P[C]P[D]$ . So

$$P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3. \quad (4)$$

In addition,  $P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6$ . To find  $P[C^c \cap D^c]$ , we first observe that

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \quad (5)$$

By De Morgan's Law,  $C^c \cap D^c = (C \cup D)^c$ . This implies

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6. \quad (6)$$

Note that a second way to find  $P[C^c \cap D^c]$  is to use the fact that if  $C$  and  $D$  are independent, then  $C^c$  and  $D^c$  are independent. Thus

$$P[C^c \cap D^c] = P[C^c]P[D^c] = (1 - P[C])(1 - P[D]) = 1/6. \quad (7)$$

Finally, since  $C$  and  $D$  are independent events,  $P[C|D] = P[C] = 1/2$ .

(d) Note that we found  $P[C \cup D] = 5/6$ . We can also use the earlier results to show

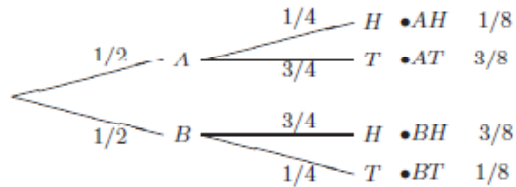
$$P[C \cup D^c] = P[C] + P[D^c] - P[C \cap D^c] = 1/2 + (1 - 2/3) - 1/6 = 2/3. \quad (8)$$

(e) By Definition 1.7, events  $C$  and  $D^c$  are independent because

$$P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C]P[D^c]. \quad (9)$$

### Problem 1.7.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \quad (1)$$

### Problem 1.8.1 Solution

There are  $2^5 = 32$  different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are  $\binom{5}{3} = 10$  codes with exactly 3 zeros.

### Problem 1.8.2 Solution

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is  $4^3 = 64$ . If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of  $4 \cdot 3 \cdot 2 = 24$  possible codes.

### Problem 1.8.4 Solution

We can break down the experiment of choosing a starting lineup into a sequence of subexperiments:

1. Choose 1 of the 10 pitchers. There are  $N_1 = \binom{10}{1} = 10$  ways to do this.
2. Choose 1 of the 15 field players to be the designated hitter (DH). There are  $N_2 = \binom{15}{1} = 15$  ways to do this.
3. Of the remaining 14 field players, choose 8 for the remaining field positions. There are  $N_3 = \binom{14}{8}$  to do this.
4. For the 9 batters (consisting of the 8 field players and the designated hitter), choose a batting lineup. There are  $N_4 = 9!$  ways to do this.

So the total number of different starting lineups when the DH is selected among the field players is

$$N = N_1 N_2 N_3 N_4 = (10)(15) \binom{14}{8} 9! = 163,459,296,000. \quad (1)$$

Note that this overestimates the number of combinations the manager must really consider because most field players can play only one or two positions. Although these constraints on the manager reduce the number of possible lineups, it typically makes the manager's job more difficult. As for the counting, we note that our count did not need to specify the positions played by the field players. Although this is an important consideration for the manager, it is not part of our counting of different lineups. In fact, the 8 nonpitching field players are allowed to switch positions at any time in the field. For example, the shortstop and second baseman could trade positions in the middle of an inning. Although the DH can go play the field, there are some complicated rules about this. Here is an excerpt from Major league Baseball Rule 6.10:

The Designated Hitter may be used defensively, continuing to bat in the same position in the batting order, but the pitcher must then bat in the place of the substituted defensive player, unless more than one substitution is made, and the manager then must designate their spots in the batting order.

If you're curious, you can find the complete rule on the web.

### Problem 1.9.1 Solution

- (a) Since the probability of a zero is 0.8, we can express the probability of the code word 00111 as 2 occurrences of a 0 and three occurrences of a 1. Therefore

$$P[00111] = (0.8)^2(0.2)^3 = 0.00512. \quad (1)$$

- (b) The probability that a code word has exactly three 1's is

$$P[\text{three 1's}] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512. \quad (2)$$

### Problem 1.9.2 Solution

Given that the probability that the Celtics win a single championship in any given year is 0.32, we can find the probability that they win 8 straight NBA championships.

$$P[8 \text{ straight championships}] = (0.32)^8 = 0.00011. \quad (1)$$

The probability that they win 10 titles in 11 years is

$$P[10 \text{ titles in 11 years}] = \binom{11}{10}(.32)^{10}(.68) = 0.00084. \quad (2)$$

The probability of each of these events is less than 1 in 1000! Given that these events took place in the relatively short fifty year history of the NBA, it should seem that these probabilities should be much higher. What the model overlooks is that the sequence of 10 titles in 11 years started when Bill Russell joined the Celtics. In the years with Russell (and a strong supporting cast) the probability of a championship was much higher.

### Problem 1.9.3 Solution

We know that the probability of a green and red light is 7/16, and that of a yellow light is 1/8. Since there are always 5 lights,  $G$ ,  $Y$ , and  $R$  obey the multinomial probability law:

$$P[G = 2, Y = 1, R = 2] = \frac{5!}{2!1!2!} \left(\frac{7}{16}\right)^2 \left(\frac{1}{8}\right) \left(\frac{7}{16}\right)^2. \quad (1)$$

The probability that the number of green lights equals the number of red lights

$$P[G = R] = P[G = 1, R = 1, Y = 3] + P[G = 2, R = 2, Y = 1] + P[G = 0, R = 0, Y = 5] \quad (2)$$

$$= \frac{5!}{1!1!3!} \left(\frac{7}{16}\right) \left(\frac{7}{16}\right) \left(\frac{1}{8}\right)^3 + \frac{5!}{2!1!2!} \left(\frac{7}{16}\right)^2 \left(\frac{7}{16}\right)^2 \left(\frac{1}{8}\right) + \frac{5!}{0!0!5!} \left(\frac{1}{8}\right)^5 \quad (3)$$

$$\approx 0.1449. \quad (4)$$